Lecture 10

Lattice Spin Systems

Plan: 1) Peierls argument for Long-range order in the Ising model with $d \geq 2$. $+1$ -1.

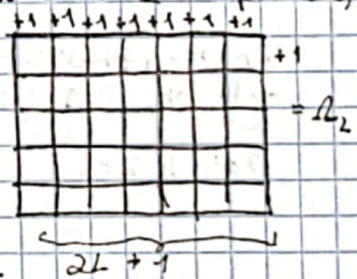
2) Mermin - Wagner thm.:

No long range order for the XY model (or the spin $O(n)$ model with $n \geq 2$) in two dimensions.

Peierls argument is applicable to discrete symmetries whereas the Mermin - Wagner thm. is relevant to continuous symmetries.

Peierls argument

We prove long-range order under $+1$ boundary cond.
(under periodic bdy. cond. there are some extra technical points - see lecture notes of P- Spinka).



$\Lambda_L := \{-L, -L+1, \dots, L\}^d$

$\partial_\circ \Lambda_L := \{ x \in \Lambda_L : \exists y \in \mathbb{Z}^d,\ y \notin \Lambda_L^{\circ}, y \sim x \}$

internal vertex boundary of Λ_L.

Configurations space of Ising:

$\Omega = \{ \sigma : \Lambda_L \to \{-1, 1\} \}$

Energy:

$$H(\sigma) = -\sum_{\substack{u,v \in \Lambda_L \\ u \sim v}} \sigma_u \sigma_v$$

Plus boundary configurations:

$$\Omega^+ = \{ \sigma \in \Omega : \begin{matrix} \sigma(v) = +1 \\ \text{for } v \in \partial_\circ \Lambda_L \end{matrix} \}$$

Plus probability measure P^+ is the prob. measure on Ω^+ with $P^+(\sigma) := \frac{1}{Z} e^{-\beta H(\sigma)}$

where Z is a normalisation constan (the partition fcn.) and $\beta = \frac{1}{T}$ is the inverse temprature parameter.
(We omit explicit mention of L).

Thm. (long-range order): In dimensions $d \geq 2$:

$$\lim_{\beta \to \infty} \limsup_{L \to \infty} \mathbb{P}^+(\sigma(0) = -1) = 0$$

$0 = (0, \dots, 0) \in \mathbb{Z}^d$.

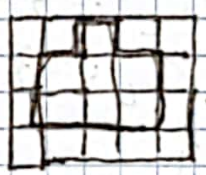
Remarks: 1) Same proof applies to other vertices $v \in \Lambda_L$

2) $\limsup_{L \to \infty}$ can also be $\sup_L$.

3) Proof showes that $P^+(\sigma(0) = -1) \leq C_1 e^{-c\beta}$ for constants $C_1, c > 0$ depending only on d.

Proof of thm:

Preliminary lemma: (Counting connected sets in a graph)
Let G be a connected graph of maximal degree Δ.
Let $v \in V(G)$. Let $\mathcal{A}_{v,k} := \{A \subseteq V(G) : \begin{array}{l} v \in A \\ A \text{ connected}, \\ |A| = k \end{array}\}$
Then, there exists $C(\Delta)$ depending only on Δ s.t. $|\mathcal{A}_{v,k}| \leq C(\Delta)^k$, for $k \geq 0$.

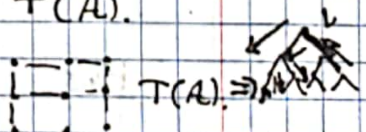


An element of $\mathcal{A}_{v,k}$ for some k.

Remark: We prove: $|\mathcal{A}_{v,k}| \leq \Delta^{2k-2}$. It is known that one can get to be $e(\Delta - 1)$ (attained for a regular tree).
maybe with a universal constant in front.

Proof of lemma:

Trick: Depth-first-Search.
To each $A \in \mathcal{A}_{v,k}$ assign in an arbitrary way, a spanning tree of A, denoted $T(A)$.
Each such tree has exactly $k-1$ edges.



It thus suffices to bound the cardinality of of $\mathcal{T}_{v,k} := \{T \subseteq G : \begin{array}{l} T \text{ is a tree with } v \in T \\ \text{and exactly } k-1 \text{ edges} \end{array}\}$

Each $T \in T_{v,k}$ may be traversed by a depth-first-search starting at v and walking exactly $2(k-1)$ steps.
(each edge is traversed twice).
There are at most $\Delta^{2(k-1)}$ such $\Delta^{2(k-1)}$ such walks.

<u>Preliminary lemma 2:</u> (Counting simply connected sets by their bdy size)

<u>Notation:</u> We consider the collection of "simply-connected" sets G in Λ_L which contain 0 and having a specific cardinality of $\partial G = \{x \in G : \exists y \in \mathbb{Z}^d, \underset{x \sim y}{} y \notin G\}$

$$C_k = \left\{ G \subseteq \Lambda_L : \begin{array}{l} 0 \in G,\ G \text{ connected} \\ \text{and } \mathbb{Z}^d \setminus G \text{ connected} \\ |\partial G| = k \end{array} \right\}$$

This requieres that $d \geq 2$.

Then $|C_k| \leq k \alpha^k$ for some α depending only on d.

Where ∂C is the edge boundary

$$\left\{ e \in E(\mathbb{Z}^d) : \begin{array}{l} e = \{u, v\} \subseteq \mathbb{Z}^d \\ u \sim v \\ \text{exactly one of } u, v \\ \text{is in } G \end{array} \right\}$$

<u>Proof:</u>
1) $G \in C_k$ is determined by ∂C (explore from 0 all vertices reachable without crossing an edge of ∂G).

2) The set of edges in $\mathbb{Z}^d$ may be endowed with a graph structure of maximal degree a fch only of d such that ∂C becomes connected for $C \in C_k$ (see Timár 2013 for a nice graph-theoretic proof).

3) Given $G \in C_k$ we may specify some $e \in \partial C$ with at most k options by picking the first edge of ∂G to the right of 0

(in the $e_1 = (1, 0, \ldots, 0)$ direction)

Consequently, $|C_k| \leq k \cdot \alpha^k$

finding some $e \in \partial G$.

applying the first lemma to ∂C.

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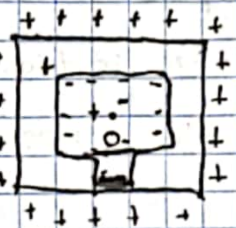
Back to the proof of the thm.:

Let $\sigma \in \Omega^+$. Suppose $\sigma(0) = -1$.
Let $G_0(\sigma)$ = The connected component of 0 in $\{v \in \Lambda_L : \sigma(v) = -1\}$.



We want a simply connected version of G_0, obtained by "adding the holes". Let G be the complement of the infinite connected component of $\mathbb{Z}^d \setminus G_0$.

The properties of $G = G(\sigma)$ are:

1) $0 \in G$
2) G and $\mathbb{Z}^d \setminus G$ are connected.
3) If $e = \{u, v\} \in \partial G$, $u \in G$, $v \notin G$ then $\sigma(u) = -1$, $\sigma(v) = +1$.

Let $\mathcal{C} = \bigcup_k \mathcal{C}_k$. Then $C(\sigma) \in \mathcal{C}$. We use a union bound:

$$\mathbb{P}^+(\sigma(0) = -1) = \mathbb{P}(C(\sigma) \neq \emptyset) \leq \sum_{k \geq 1} \sum_{G \in \mathcal{C}_k} \mathbb{P}^+(C(\sigma) = G)$$

We now use the discrete symmetry to bound

$$\mathbb{P}^+(C(\sigma) = C) = \frac{\sum_{\substack{\sigma \in \Omega^+ \\ C(\sigma) = G}} e^{-\beta H(\sigma)}}{\sum_{\sigma \in \Omega^+} e^{-\beta H(\sigma)}} \leq \frac{\sum_{\substack{\sigma \in \Omega^+ \\ G(\sigma) = C}} e^{-\beta H(\sigma)}}{\sum_{\substack{\sigma \in \Omega^+ \\ C(\sigma) = C}} e^{-\beta H(\sigma^G)}}$$

This is the partition fch. Z.

$$= e^{-2\beta |\partial G|}.$$

For each $\sigma \in \Omega^+$ define $\sigma^G \in \Omega^+$

$$\sigma^G(v) = \begin{cases} \sigma(v) & v \notin G \\ -\sigma(v) & v \in G \end{cases}$$

Note, if $G(\sigma) = C$ then $H(\sigma) - H(\sigma^G) = 2|\partial G|$. The map $\sigma \to \sigma^G$ is one to one.

Substituting back, $\mathbb{P}^+(\sigma(0) = -1) \leq \sum_{k \geq 1} \sum_{G \in \mathcal{C}_k} \mathbb{P}^+(G(\sigma) = G)$

$$\leq \sum_{k \geq 1} \sum_{G \in \mathcal{C}_k} e^{-2\beta \underbrace{|\partial G|}_{=k}} = \sum_{k \geq 1} e^{-2\beta k} |\mathcal{C}_k| \leq$$

$$\leq \sum_{k \geq 1} e^{-2\beta k} \cdot k \cdot d^k \leq C e^{-c\beta}$$

Lemma 2

for β sufficiently large as a function of d, for constants $C, c > 0$ depending on d.

Mermin - Wagner thm.

$$\Lambda_L := \{-L, -L+1, \dots, L\}^d$$

$$\Omega = \{\sigma : \Lambda_L \to S^1\}$$ XY model spins are on ⊙

$$H(\sigma) := \sum_{\substack{u,v \in \Lambda_L \\ u \sim v}} \sigma_u \cdot \sigma_v$$ inner product in $\mathbb{R}^2$

$$\Omega^+ := \{\sigma \in \Omega : \begin{array}{l} \sigma(v) = e_1 = (1,0) \\ \text{for } v \in \partial_\circ \Lambda_L \end{array}\}$$

$\mathbb{P}^+$ is the measure on Ω^+ whose density is $\frac{1}{Z} e^{-\beta H(\sigma)}$ with respect to $\prod_{v \in \Lambda_L \setminus \partial_\circ \Lambda_L} \lambda(\sigma(v)) =: d\sigma$

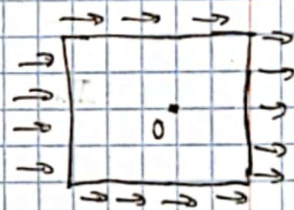
where λ is Lebesgue measure on S^1.

Thm. (no long range order in two dimensions):

In $d=2$, for any $0 < \beta < \infty$,

$$\lim_{L \to \infty} \| \mathbb{E}^+ (\sigma(0)) \| = 0.$$

Euclidean norm

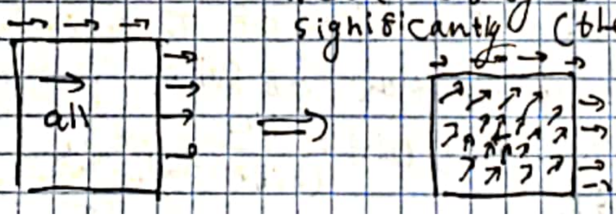


Remarks: 1) This works also for other vertices as long as $\|v\|_1 = o(L)$ as $L \to \infty$.

2) The proof actually shows that the density of $\sigma(0)$ tends to the uniform density as $L \to \infty$.

3) The proof does not show that there is a unique Gibbs measure.
It is known that there is a unique translation-invariant Gibbs measure but unknown that there is a unique Gibbs measure.
(Can the bdy values significantly alter the joint dist. at a finite set of vertices near the origin?) i.e. uniformly in L.

Idea of proof: There are low-energy excitations which modify the spin $\sigma(0)$ significantly (the spin wave).



Energetic difference of these two configurations is only $\frac{c}{\log L}$ if the rotation is faster near center.

<u>Remark:</u> Quantitatively, $\|\mathbb{E}\,\sigma(0)\| \leq \frac{C_1(\beta)}{L^{c(\beta)}}$

but we will only show:

$$\|\mathbb{E}\,\sigma(0)\| \leq \frac{C_1(\beta)}{\log L}.$$

How to apply this idea?

We seen a way of perturbing a given configuration $\sigma \in \Omega^+$ which, on the one hand, does not modify the density of σ much but, on the other hand, modifies $\sigma(0)$ significantly.

We explain it in the XY context but the strategy is widely applicable.

<u>Spin wave:</u> Regard $\sigma \in \Omega^+$ as a vector of angles: $\sigma(v) = e^{i\theta_v}$ where $\theta_v \in \mathbb{R}/_{2\pi\mathbb{Z}}$ ← $[0,2\pi)$ regarded periodic modulo 2π.

In these terms, density of $\theta: \Lambda_L \to \mathbb{R}/_{2\pi\mathbb{Z}}$ is

$$\frac{1}{Z}\, e^{\beta \sum_{u\sim v} \cos(\theta_u - \theta_v)}$$

with respect to product Lebesgue measure $d\theta$.

Given $\theta \in \Omega^+ = \{\theta : \Lambda_L \to \mathbb{R}/_{2\pi\mathbb{Z}} : \theta_v = 0 \text{ for } v \in \partial_\circ \Lambda_L\}$

We will consider both $\theta + \tau$ and $\theta - \tau$ for some $\tau \in \Omega^+$.

<u>Key calculation:</u> $\sqrt{\rho(\theta+\tau)\,\rho(\theta-\tau)} \geq \rho(\theta)\, e^{-C\sum_{u\sim v}(\tau(u)-\tau(v))^2}$

for some absolute constant $C>0$.

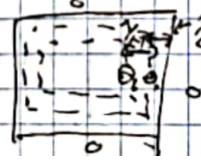
That is for any $\theta, \tau \in \Omega^+$.

<u>Fact:</u> In $d=2$ there exists $\tau \in \Omega^+$ with $\tau(0) = \pi$ and $\sum_{u\sim v} (\tau(u) - \tau(v))^2 \leq \frac{C_1}{\log L}$ for an absolute constant $C_1 > 0$.

<u>proof:</u> (of Fact): One choice of τ is such that $\tau(v)$ is a function of $\|v\|_1$.

with $\tau(\|v\|) - \tau(\|v\|+1) = \frac{c(L)}{(\|v\|+1)\cdot \log L}$

$c(L) \xrightarrow{L\to\infty} \pi$.



rotation $\frac{c(L)}{r \log L}$ at radius r.

We get $\sum_{u \sim v} (\tau(u) - \tau(v))^2 \leq \sum_{k=1}^{L} \frac{c(L)^2}{k^2 \log^2 L} \cdot 4k$

$= \sum_{k=1}^{L} \frac{4c(L)^2}{k \log^2 L} \leq \frac{C_8}{\log L}.$